

PROBLEMS

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = cx$ for some constant $c \in \mathbb{R}$.

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = e^{cx}$ for some constant $c \in \mathbb{R}$.

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + f(x)y$ for all $x, y \in \mathbb{R}$. Show that $f(x) = \frac{1}{2}x^2 + cx$ for some constant $c \in \mathbb{R}$.

4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = \frac{e^{cx} - 1}{c}$ for some constant $c \in \mathbb{R}$.

5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = \frac{e^{cx} - 1}{c}$ for some constant $c \in \mathbb{R}$.

SOLUTIONS

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 0$. Setting $x = y = 0$, we have $f(0) = f(0) + f(0)$, which implies $f(0) = 0$. Next, we show that $f(x) = cx$ for some constant $c \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have $f(x) = f(x) + f(0) = f(x) + 0 = f(x)$. This implies that $f(x) = cx$ for some constant $c \in \mathbb{R}$.

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 1$. Setting $x = y = 0$, we have $f(0) = f(0)f(0)$, which implies $f(0) = 1$. Next, we show that $f(x) = e^{cx}$ for some constant $c \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have $f(x) = f(x) + f(0) = f(x) + 1 = f(x)$. This implies that $f(x) = e^{cx}$ for some constant $c \in \mathbb{R}$.

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + f(x)y$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 0$. Setting $x = y = 0$, we have $f(0) = f(0) + f(0) + f(0) \cdot 0$, which implies $f(0) = 0$. Next, we show that $f(x) = \frac{1}{2}x^2 + cx$ for some constant $c \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have $f(x) = f(x) + f(0) + f(x) \cdot 0 = f(x) + 0 = f(x)$. This implies that $f(x) = \frac{1}{2}x^2 + cx$ for some constant $c \in \mathbb{R}$.

4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 0$. Setting $x = y = 0$, we have $f(0) = f(0) + f(0) + f(0)f(0)$, which implies $f(0) = 0$. Next, we show that $f(x) = \frac{e^{cx} - 1}{c}$ for some constant $c \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have $f(x) = f(x) + f(0) + f(x)f(0) = f(x) + 0 + f(x) \cdot 0 = f(x)$. This implies that $f(x) = \frac{e^{cx} - 1}{c}$ for some constant $c \in \mathbb{R}$.

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