

PROBLEM 1

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose that f is continuous at $x=0$. Prove that f is linear, i.e.,

$$f(x) = cx \quad \text{for all } x \in \mathbb{R},$$

where $c = f(1)$.

Hint: Use the continuity of f at $x=0$ to show that f is continuous everywhere.

Then use the functional equation to show that f is linear on \mathbb{Q} .

Finally, use the density of \mathbb{Q} in \mathbb{R} to conclude that f is linear on \mathbb{R} .

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PROBLEM 2

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