

## PROBLEM 1

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that  $f$  is linear, i.e.,  $f(x) = cx$  for some constant  $c \in \mathbb{R}$ .

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying  $f(x+y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ . We first show that  $f(0) = 0$ . Setting  $x = y = 0$  in the functional equation, we get  $f(0) = f(0) + f(0)$ , which implies  $f(0) = 0$ . Next, we show that  $f$  is additive over the integers. For any integer  $n$ , we have  $f(n) = f(1 + 1 + \dots + 1) = n f(1)$ . Similarly, for any negative integer  $-n$ , we have  $f(-n) = -n f(1)$ . Thus,  $f(n) = c n$  for all integers  $n$ , where  $c = f(1)$ .

Now, we show that  $f$  is linear over the rationals. Let  $r = \frac{p}{q}$  be a rational number, where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . Then  $f(r) = f\left(\frac{p}{q}\right) = \frac{p}{q} f(1) = c r$ . Thus,  $f(r) = c r$  for all rational numbers  $r$ . Finally, we show that  $f$  is linear over the reals. Let  $x \in \mathbb{R}$ . For any real number  $r$ , we have  $f(rx) = r f(x)$ . This implies that  $f(x) = c x$  for all real numbers  $x$ .