

PROBLEMS

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = cx$ for some constant $c \in \mathbb{R}$.

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = e^{cx}$ for some constant $c \in \mathbb{R}$.

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + cxy$ for all $x, y \in \mathbb{R}$, where $c \in \mathbb{R}$. Show that $f(x) = \frac{c}{2}x^2 + dx$ for some constant $d \in \mathbb{R}$.

4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + cxy$ for all $x, y \in \mathbb{R}$, where $c \in \mathbb{R}$. Show that $f(x) = \frac{c}{2}x^2 + dx$ for some constant $d \in \mathbb{R}$.

SOLUTIONS

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 0$. Let $x = y = 0$, then $f(0) = f(0) + f(0)$, which implies $f(0) = 0$. Next, we show that $f(x) = cx$ for some constant $c \in \mathbb{R}$. Let $x = 1$, then $f(1) = c$. For any $x \in \mathbb{R}$, we have $f(x) = f(x \cdot 1) = xf(1) = cx$.

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 1$. Let $x = y = 0$, then $f(0) = f(0)f(0)$, which implies $f(0) = 1$. Next, we show that $f(x) = e^{cx}$ for some constant $c \in \mathbb{R}$. Let $x = 1$, then $f(1) = c$. For any $x \in \mathbb{R}$, we have $f(x) = e^{cx}$.

3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + cxy$ for all $x, y \in \mathbb{R}$, where $c \in \mathbb{R}$. We first show that $f(0) = 0$. Let $x = y = 0$, then $f(0) = f(0) + f(0) + c \cdot 0 \cdot 0$, which implies $f(0) = 0$. Next, we show that $f(x) = \frac{c}{2}x^2 + dx$ for some constant $d \in \mathbb{R}$. Let $x = 1$, then $f(1) = d + \frac{c}{2}$. For any $x \in \mathbb{R}$, we have $f(x) = \frac{c}{2}x^2 + dx$.

4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y) + cxy$ for all $x, y \in \mathbb{R}$, where $c \in \mathbb{R}$. We first show that $f(0) = 0$. Let $x = y = 0$, then $f(0) = f(0) + f(0) + c \cdot 0 \cdot 0$, which implies $f(0) = 0$. Next, we show that $f(x) = \frac{c}{2}x^2 + dx$ for some constant $d \in \mathbb{R}$. Let $x = 1$, then $f(1) = d + \frac{c}{2}$. For any $x \in \mathbb{R}$, we have $f(x) = \frac{c}{2}x^2 + dx$.

LINGSTON

