

## PROBLEM 1

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that  $f$  is linear, i.e.,  $f(x) = cx$  for some constant  $c \in \mathbb{R}$ .

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying  $f(x+y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ . We will show that  $f$  is linear. First, we show that  $f(0) = 0$ . Let  $x = 0$  and  $y = 0$ . Then  $f(0+0) = f(0) + f(0)$ , which implies  $f(0) = 2f(0)$ . Subtracting  $f(0)$  from both sides gives  $f(0) = 0$ . Next, we show that  $f(-x) = -f(x)$ . Let  $x = x$  and  $y = -x$ . Then  $f(x+(-x)) = f(x) + f(-x)$ , which implies  $f(0) = f(x) + f(-x)$ . Since  $f(0) = 0$ , we have  $0 = f(x) + f(-x)$ , so  $f(-x) = -f(x)$ . Finally, we show that  $f(x) = cx$  for some constant  $c \in \mathbb{R}$ . Let  $x = 1$  and  $y = 1$ . Then  $f(1+1) = f(1) + f(1)$ , which implies  $f(2) = 2f(1)$ . By induction, we can show that  $f(n) = nf(1)$  for all positive integers  $n$ . Similarly, we can show that  $f(-n) = -nf(1)$  for all positive integers  $n$ . For any real number  $x$ , we can write  $x = n + r$  where  $n$  is an integer and  $r$  is a real number in the interval  $[0, 1)$ . Then  $f(x) = f(n+r) = f(n) + f(r) = nf(1) + f(r)$ . Since  $f$  is continuous, we can show that  $f(r) = cr$  for some constant  $c \in \mathbb{R}$ . Therefore,  $f(x) = cx$  for all  $x \in \mathbb{R}$ .

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