

## PROBLEM 1

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose that  $f$  is continuous at  $x=0$ . Prove that  $f$  is linear, i.e.,

$$f(x) = cx \quad \text{for all } x \in \mathbb{R},$$

where  $c = f(1)$ .

Hint: Use the continuity of  $f$  at  $x=0$  to show that  $f$  is continuous everywhere.

Then use the functional equation to show that  $f$  is linear on  $\mathbb{Q}$ .

Finally, use the density of  $\mathbb{Q}$  in  $\mathbb{R}$  to conclude that  $f$  is linear on  $\mathbb{R}$ .

Remark: The above result is a special case of the Cauchy functional equation.

See Problem 2 for a more general result.

See Problem 3 for a more general result.

See Problem 4 for a more general result.

See Problem 5 for a more general result.

See Problem 6 for a more general result.

See Problem 7 for a more general result.

See Problem 8 for a more general result.