

PROBLEM 1

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that f is linear, i.e., $f(x) = cx$ for some constant $c \in \mathbb{R}$.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. We first show that $f(0) = 0$. Setting $x = y = 0$ in the functional equation, we get $f(0) = f(0) + f(0)$, which implies $f(0) = 0$. Next, we show that f is additive over the integers. For any integer n , we have $f(n) = f(1 + 1 + \dots + 1) = n f(1)$. Similarly, for any negative integer $-n$, we have $f(-n) = -n f(1)$. Thus, $f(n) = c n$ for all integers n , where $c = f(1)$.

Now, we show that f is linear over the rationals. Let $r = \frac{p}{q}$ be a rational number, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Then $f(r) = f\left(\frac{p}{q}\right) = \frac{p}{q} f(1) = c r$. Thus, $f(r) = c r$ for all rational numbers r . Finally, we show that f is linear over the reals. Let $x \in \mathbb{R}$. For any real number $\epsilon > 0$, there exists a rational number r such that $|x - r| < \epsilon$. Then $|f(x) - f(r)| = |f(x - r)| = |f(x - r)|$. Since f is additive, we have $|f(x - r)| = |f(x) - f(r)|$. Thus, $|f(x) - c x| = |f(x - r) - c(x - r)| = |f(x - r) - c(x - r)|$. Since f is additive, we have $|f(x - r) - c(x - r)| = |f(x - r) - c(x - r)|$. Thus, $|f(x) - c x| < \epsilon$. Since ϵ is arbitrary, we have $f(x) = c x$ for all real numbers x .