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Introduction

Algebra is an English noun of Latin and Arabic origin. The original sense was "the surgical treatment of fractures," so in mathematics, it refers to restoring what is missing and equating like with like. Thus, our study of algebra will proceed according to this original sense of equating, or relating, numerical expressions and "restoring what is missing" by determining the numerical value of unknown quantities in relations.

The classical discipline of algebra is **the branch of mathematics in which arithmetical operations and formal manipulations are applied to abstract symbols rather than specific numbers**. You are proficient at applying arithmetical operations to numbers and evaluating numerical expressions, but algebra expands operations to include expressions containing one or more unknown quantities and expands the evaluation of expressions to include solving equations for unknown quantities. The key difference between arithmetic and algebra is the unknown quantity and the necessary relation to determine unknown quantities: **the equation**.

As a branch of mathematics, you may be tempted to relegate algebra to the science, technology, engineering, and math fields (STEM). But until the early twentieth century, mathematics was a discipline of the liberal arts! We can observe humans in history utilizing numbers to describe the world as early as around 2000 B.C. among the Babylonian and Mesopotamian civilizations. If we understand philosophy to be the study of the fundamental nature of knowledge, reality, and existence, then mathematics is an integral part of our pursuit of knowledge of the cosmos. This is precisely why the first true mathematicians were philosophers in ancient Greece! In fact, one of the earliest Greek philosophers, Pythagoras (570-490 B.C.), and his student Plato (428-347 B.C.), argued that the whole of the world is mathematical in nature, so the pursuit of true knowledge begins with a thorough understanding of mathematics. Thus, in the interest of a classical rather than vocational education, this course will pursue mastery of algebraic concepts as a complementary discipline oriented in the classical liberal arts rather than as a disjointed STEM discipline. Whether you want to be an engineer or not, algebra is an integral and formative part of your classical education.

Your study of algebra will proceed according to two basic premises:

1. **Algebra is the study of relationships.** Just as geometry focuses on spatial concepts and calculus is concerned with rates of change, algebra is a comprehensive exploration of mathematical relationships, including both equations and inequalities.

The most basic notion in algebra, in accordance with its etymology (see above), is the **equation**. An equation is a formal statement that two mathematical expressions are equal. The equation will form the foundation for your study of algebra.

2. **Algebraic concepts develop by degrees.** Thus, first-degree (linear) relations will be mastered before moving on to second-degree (quadratic) relations and higher, rational degree (rational and radical) relations — ultimately culminating in relations with variable degrees (exponential and logarithmic functions). You will master all aspects of each of these degrees before moving on to the next, and you will do so with a **consistent methodology** from beginning to end in order to pursue maximum integration and comprehension. Simply put, students should carry out the same operations and computational techniques the same way every time.

Therefore, the logical scope of your study of the discipline of algebra will cover mathematical relations of all degrees (premise 1). The logical sequence of concepts will begin with a mastery of first-degree relations and grow systematically to include increasingly more sophisticated degrees (premise 2). The general flow of each unit is based on the logical introduction of any new mathematical concept:

1. Define the new concept and describe it in detail.
2. Explore operations involving the new concept.
3. Examine relations involving the new concept and develop strategies to resolve them.

THE HISTORY OF ALGEBRA

In an effort to orient your study of algebra both in traditional Western liberal arts and philosophy and to understand how the discipline of algebra developed over time, you will familiarize yourself with the historical development of the concepts that form the foundation of algebra throughout this course.

The discipline of algebra developed as mathematicians pursued a consistent, systematic, and coherent method of relating mathematical expressions and operations. Many modern-day processes would not be possible apart from mathematicians striving to solve this problem. This pursuit ultimately arrived at the concept of the equation. The consolidation of the concept of the equation followed **three main threads** as they developed in the ancient Middle East and Greece, the Islamic era, and the European Renaissance:

1. Attempts to solve equations involving one or more unknown quantities.
2. The evolution of what qualifies as a legitimate number. Broader domains flexible enough to support the abstract structure of symbolic algebra were required. Simply put, philosophers as early as Pythagoras discovered that whole numbers were insufficient to account for the mathematical possibilities that they were observing in creation.
3. Refinement of a symbolic language suitable for devising, conveying, and solving entire categories of mathematical problems. Believe it or not, until the advent of our modern algebraic notation by René Descartes (1596-1650), all math was word problems!

Thus, you will find historical introductions at key points throughout the sequence of the course as your concept of what qualifies as a legitimate number grows, as you learn the proper symbolic language, or vocabulary, with which to engage algebra, and as you learn to solve equations of increasing degrees and with one or more unknown variables.

ALGEBRA II

This course is the third and final course in the Memoria Algebra Series. The necessary prerequisite for this course is *Memoria Algebra I*. Because the scope and sequence of the Memoria Algebra Series intentionally departs from the modern convention, it is very difficult to transition from a more conventional Algebra I course into *Memoria Algebra II*. If you have questions about prerequisites and readiness, please contact Memoria Press. In short, if you have mastered operations and solving methodologies with two- and three-variable linear systems, polynomials, rational expressions and relations, and radical expressions and relations, and if you have been introduced to the set of complex numbers and operations with complex numbers, you will be successful in *Memoria Algebra II*. The purpose of this course is to build on the conceptual and computational foundation laid in *Memoria Pre-Algebra* and synthesize the complex computational skills learned in *Memoria Algebra I*. While problem-solving methodology remains an important aspect of this course, the focus shifts to graphical representations of complex, higher-degree solution sets in the Cartesian coordinate system. Although the final unit in *Memoria Algebra II* covers exponential and logarithmic relations, the entire Memoria Algebra Series culminates in Unit III with the fundamental theorem of algebra. Be prepared to match the logical and computational rigor of algebraic problem solving with your own rigor and diligence in memorization and recitation, exercise completion and correction, and successful completion of the assessments. This course, and the entire Memoria Algebra Series, is the necessary prerequisite for *Memoria Trigonometry*.

CURRICULUM COMPONENTS

The *Memoria Algebra II* curriculum consists of these complementary tools:

1. *Student Text*
2. *Teacher Manual*
3. *Solutions Key*
4. *Quizzes & Tests*
5. Instructional Videos

The instructional videos are intended to serve as a supplement to the written curriculum and are not required for success in the course. The co-author of this curriculum taught this material to a class of students at Highlands Latin Cottage School in Louisville, Kentucky. The class met once weekly for ninety minutes, so some lessons received more attention than others due to the time limitations. You will find that the lesson notes in the *Teacher Manual* match the presentation of the material in the videos closely. Use the videos to reinforce lesson content as desired, especially when students struggle with certain concepts.

The primary features of the *Student Text*, besides providing problem banks with which students can work toward mastery, include clear statements of lesson objectives, introduction and review of important terms, and detailed consideration of example problems. Ensure that students master the terms and utilize the examples as the main instructional tool as they move toward completing exercises independently. The first couple exercises can be used for further demonstration as required. All exercises should be completed, checked for accuracy, and corrected for each lesson, and problematic exercises can be repeated for mastery. An answer key for the odd exercises is included in the Appendix. Ensure that students have met the lesson objective before moving on.

The *Quizzes & Tests* evaluate student progress. Often open-response questions are utilized that require students to state, in writing, their understanding of the concept under examination. This is an invaluable skill that ensures that students understand the mathematical concepts they are learning rather than becoming mere computers. Throughout this course, there are quizzes at the conclusion of each part and tests at the conclusion of each unit. Quizzes focus on the material in their respective parts, but exams are cumulative. There is also a Final Exam that should be attempted after intensive review. Mastery of the material presented in this course should translate to a score of 85 or greater on the Final Exam. Reserve Form B assessments for retakes or for Final Exam review in the closing weeks of the course.

The *Teacher Manual* serves as the primary instructional resource for the course and includes an answer key for all lesson exercises. Before presenting a lesson to students, thoroughly review the material and the teacher notes for helpful techniques and emphases. A teacher or homeschool instructor with an elementary understanding of algebra will be able to convey the lesson concepts and achieve the lesson objectives by understanding and following the material included in the *Teacher Manual*. The *Solutions Key* contains complete, detailed solutions for every exercise to ensure comprehension.

A recitation schedule is located in the Appendix of the *Student Text* and *Teacher Manual*. Students should master the responses to the recitation questions for each lesson, reviewing recitation questions as necessary until they are mastered.

PROBATIO

Exponents, Polynomials, Solving Equations, and the Cartesian Coordinate System

The purpose of this introductory unit is to revisit, refresh, and review several key concepts and computational skills that you mastered in *Memoria Algebra I*: exponential properties and operations, polynomial special products and factoring, the Cartesian coordinate system, systems of linear equations, rational and radical relations, graphing functions, and word problem analysis. This will ensure that you build on a solid foundation as you launch into the content of this course.

This unit is titled *Probatio* to emphasize its second and complementary purpose—to *test* and *prove* that you have mastered the concepts and skills necessary for success in *Memoria Algebra II*. In lieu of a readiness test, these lessons, along with their exercises and the *Probatio* Test, will indicate your preparedness for the content of *Memoria Algebra II* and instill the confidence necessary for you to master it.

Algebra is a cumulative discipline. You are about to embark on the third and final year of your mastery of algebra. Before you do, it is necessary to shore up the concepts and skills that you learned in the first two years. Keep this in mind as you work through these first 11 lessons.

PROBATIO LESSON 1

Prior to beginning work on this lesson, read the *Probatio* introduction together with students.

Students learned the content of this lesson in *Memoria Pre-Algebra* and *Algebra I*, but it is an important and necessary place to start and foundational information for the next two lessons. This lesson should be completed in one day.

Instruct students to learn the Important Terms. Ensure that they can identify the base and exponent of various powers. Always use the proper terminology when you read powers.

Example 1a should solidify exponential notation as a shorthand expression for multiplication with like factors. In Example 1b, emphasize the importance of considering groups of like factors in an expression. Consider the fours, the x 's, and the y 's separately. The expression becomes the product of three different powers. Remind students that different numerals and/or variables written adjacent to each other indicate the multiplication operation.

Example 2a is brief. It covers the method for evaluating simple arithmetic exponents, namely, a positive integer base raised to a whole number exponent. Convert powers to products and evaluate. Use a calculator if the product is too large or unfamiliar. Example 2b introduces the complexities involving the negative symbol, which is a terrific opportunity to focus on base identification with grouping symbols present. Take the time here to show students the difference between -2^5 and $-(2)^5$ even though they both have the same value.

You will refresh arithmetic order of operations in Example 3. Remind students of the PEMDAS memory aid, which stands for (P)arentheses, or grouping symbols; (E)xponents; (M)ultiplication and (D)ivision; and (A)ddition and (S)ubtraction. Multiplication and division operations can be evaluated

Probatio Lesson 1: Exponential Notation

OBJECTIVE: Define exponential notation, properly identify exponential bases, and evaluate powers with arithmetic order of operations.

IMPORTANT TERMS

1. **Exponent Notation:** An efficient mode of expression for multiplication in which all the factors are the same. The factor is written only once, and a small numeral indicating how many times that factor appears is written as a superscript to the right of the factor.
2. **Base:** In exponent notation, the factor being multiplied.
3. **Exponent:** In exponent notation, the superscript number to the right of the base that indicates how many times to use the base as a factor.
4. **Power:** An alternative expression for exponent notation. For example, we read the power 3^4 , which is $3 \cdot 3 \cdot 3 \cdot 3$ in exponent notation, as "three to the fourth power."

EXAMPLE 1

Write each of the following expressions in exponential notation.

- a. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ b. $4 \cdot 4 \cdot x \cdot x \cdot x \cdot y \cdot y$

Solution

- a. The factor 3 appears five times in the product. Write the base, 3, and then the number of times it appears as a factor in the product, 5, in superscript to the right of the base.

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5$$

- b. The factor 4 appears twice, the factor x appears three times, and the factor y appears twice. Write each base factor with its number of appearances to the right in superscript.

$$4 \cdot 4 \cdot x \cdot x \cdot x \cdot y \cdot y = 4^2 \cdot x^3 \cdot y^2 = (4^2)x^3y^2$$

The fully simplified open phrase would be $16x^3y^2$.

Note: Evaluating powers accurately depends on your mastery of base identification. In Example 1a, the base of the power is 3. In Example 1b, on the other hand, there are three different bases: 4, x , and y , and each one has its own exponent determined by how many times it appears as a factor in the product. The fully simplified open phrase is the product of three exponential factors.

In general, if there are no grouping symbols present, the base of a power is the numeral or variable immediately to the left of the exponent. If the base of a power is a group with multiple factors (see Example 2b), the base includes everything inside the group.

one at a time from left to right in the expression; multiplication does not need to be evaluated before division. The same is true for addition and subtraction, respectively. This is because division is multiplication of the reciprocal and subtraction is addition of the opposite. In other words, division and subtraction are derivative expressions of multiplication and addition, respectively. Thus, the order of operations applies to multiplication and addition. Multiplication operations must be evaluated before addition operations.

There are three sets of exercises for this lesson, corresponding to the three different examples. Students should complete all exercises for this lesson on the same day you teach it. Check student work for accuracy, then instruct them to correct any errors themselves. If they are unable to do so, try to help them along in such a way so that you are not telling them the answers. Only consult the *Solutions Key* if there is a significant gap in understanding.

EXAMPLE 2

Evaluate each of the following powers.

a. 5^3

b. -2^5

Solution

- a. Convert the given power into a product to evaluate.

$$5^3 = 5 \cdot 5 \cdot 5 = 25 \cdot 5 = 125$$

- b. This closed phrase reads, "the opposite of the fifth power of 2." To evaluate this phrase correctly, compute the fifth power of 2 and then take its opposite by multiplying it by -1 .

$$-2^5 = -(2^5) = -1 \cdot 2^5 = -(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = -(32) = -32$$

Note: If the phrase in Example 2b had included grouping symbols as shown below, the base of the power would be -2 rather than 2 .

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$$

This expression reads, "the opposite of 2 raised to the fifth power," or "the fifth power of negative 2." The grouping symbol indicates that the base of the power includes two factors: 2 and -1 .

$$(-2)^5 = (-1 \cdot 2)^5$$

Finally, recall that when multiplying real numbers, an odd number of negative factors results in a negative product, while an even number of negative factors results in a positive product.

EXAMPLE 3

Evaluate the following closed phrase according to arithmetic order of operations.

$$6 + 5(7 - 3)^2 - 2^4$$

Solution

Recall that per arithmetic order of operations, you must evaluate operations within grouping symbols first, then evaluate all exponents, then evaluate all multiplication and division operations one at a time from left to right, and finally evaluate all addition and subtraction operations one at a time from left to right.

$$6 + 5(7 - 3)^2 - 2^4$$

$$= 6 + 5(4)^2 - 2^4$$

$$= 6 + 5(16) - 2^4$$

$$= 6 + 80 - 16$$

$$= 86 - 16$$

$$= 70$$

EXERCISE KEY

- 1. 4^3
- 2. $(-3)^4$
- 3. n^5
- 4. $4^3r^2t^3$
- 5. 7^2
- 6. 6^5
- 7. 3^4a^4
- 8. 0^5
- 9. $5^2c^3d^4$
- 10. -64
- 11. 3125
- 12. 1296
- 13. 64
- 14. -64
- 15. -64
- 16. 324
- 17. -36
- 18. 432
- 19. 512
- 20. 18
- 21. -10
- 22. 18
- 23. 10100
- 24. 347
- 25. -44
- 26. 58
- 27. -75
- 28. 45
- 29. 2

EXERCISES

Write each of the following products in exponential notation.

1. $4 \cdot 4 \cdot 4$

4. $4 \cdot 4 \cdot 4 \cdot r \cdot r \cdot t \cdot t \cdot t$

7. $3 \cdot a \cdot 3 \cdot a \cdot 3 \cdot a \cdot 3 \cdot a$

2. $(-3) \cdot (-3) \cdot (-3) \cdot (-3)$

5. $7 \cdot 7$

8. $0 \cdot 0 \cdot 0 \cdot 0 \cdot 0$

3. $n \cdot n \cdot n \cdot n \cdot n$

6. $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6$

9. $5 \cdot 5 \cdot c \cdot c \cdot c \cdot d \cdot d \cdot d \cdot d$

Evaluate each of the following powers.

10. -8^2

12. 6^4

14. -4^3

16. $2^2 \cdot 3^4$

18. $4^2 \cdot 3^3$

11. 5^5

13. $(-8)^2$

15. $(-4)^3$

17. $-2^2 \cdot 3^2$

19. $(-4)^3 \cdot (-2)^3$

Evaluate each of the following closed phrases according to arithmetic order of operations.

20. $3^2 \cdot 5 - 2^3 + 75 \div 15$

25. $-2[10 - 3(4 - 2^3)]$

21. $(3 \cdot 5^2 - 15 \div 3) \div (-7)$

26. $4 + 6[(9 - 3)^2 + (2^2 - 7)^3]$

22. $2 \cdot (2^3 - 5) \cdot 3$

27. $2^2(3^2 - 2 \cdot 2^3) - (20 + 3^3)$

23. $(-10)^2 - 5(10^3) \cdot (-2)$

28. $3 + 2[4(12 - 7) + (2^3 - 3^2)^2]$

24. $7(5 \cdot 2 - 3^2) + 5[6^2 + (2)^8]$

29. $\frac{-2(15 - 3 \cdot 2^2)}{5^2 - 3^2} + \frac{11 + 12 \div 2^2 \cdot 3^2}{2(13 - 5)}$

UNIT I

Higher-Degree Relations – Quadratic Functions and Transformations

Having reviewed foundational concepts and computational skills from *Memoria Algebra I*, you are prepared to focus on a subset of polynomial functions – quadratic functions. A quadratic function is one in which the independent variable is second degree while the dependent variable is first degree. You already know quadratic standard form, and the *parent* quadratic function is $y = x^2$. In this unit, you will learn how to solve quadratic functions, how to graph them, and how to compare and contrast the graphs of quadratic functions to the parent function. The graph of any quadratic function is a transformation of the parent function.

Mastering quadratic functions algebraically and graphically is an application of the discipline that we call *analytical geometry* today. This correlation between algebraic symbolism and geometric symbolism to describe or model observed mathematical features of the universe was pioneered by Pierre de Fermat and René Descartes. Listen to Descartes explain the occasion and necessity for his innovation:

"I have resolved to quit only abstract geometry, that is to say, the consideration of questions which serve only to exercise the mind, and this, in order to study *another kind of geometry*, which has for its object the explanation of the phenomena of nature."

Revolutions in scientific knowledge at the time – namely, the heliocentric theory of Copernicus, the motion of ships at sea, and an increased interest in the study of light – demanded a more practical, concrete application of the abstract geometry of Euclid. By using both polynomial functions and the parabola to model, for instance, objects in motion under the influence of the force of gravity, mathematicians and scientists were able to significantly advance human understanding of the physical universe. Your study of quadratic functions and transformations in this unit is your first significant step into this rigorous, exciting foray.

PART A: SOLVING RELATIONS IN QUADRATIC STANDARD FORM

The purpose of these lessons is to learn how to determine the solutions for any quadratic function. The solutions of a quadratic function are also called roots or zeros and correspond to the x -intercepts in the Cartesian coordinate system. In short, they are the values of x that result in a zero value for y .

You will learn the nuances for solving these equations when either of the coefficients b or c in quadratic standard form are zero. Then you will learn the basic solving methodology called completing the square and its derivative, the quadratic formula. The quadratic formula will enable you to determine the roots – real and/or complex – of any quadratic function. You will also learn how to write a solution set for a quadratic inequality and graph it on a one-dimensional number line.

LESSON 12

Introduce the quadratic relation as a special subset of higher-degree polynomial relations. Quadratic relations are second degree. Also review quadratic standard form (*Memoria Algebra I*, Unit V, Lesson 91, Multiplying Complex Numbers) and ensure students can define it and recite the generalization. This is especially relevant in this lesson as each exercise requires students to identify all three coefficients. The a coefficient is always associated with the quadratic, or second-degree, term; the b coefficient is always the coefficient of the linear, or first-degree, term; and the c coefficient is always the constant term. It is important in this lesson to habituate students to always convert second-degree relations into quadratic standard form. It is also important to habituate them to identify the three coefficients. This will be especially relevant when they learn the quadratic formula in Lesson 14.

The three parts of Example 1 correspond to the three possibilities in this lesson: In each case, one of the coefficients is zero. When $a = 0$, the relation is no longer quadratic as the second-degree term is eliminated. When $b = 0$ or $c = 0$, the results can be generalized as each of those examples shows. Students will need to utilize the general factoring strategy and the zero-product property to solve for x in these situations. Helping students to see the expected result in each case will provide a helpful accountability check when they are completing the exercises. Ensure students are expressing solution sets in set notation.

Finally, a note on "roots" (see Important Term #3). The lessons in Part C of this unit will introduce students to graphing quadratic functions. When they do, they will find that the algebraic "solution(s)" or "root(s)" for a quadratic equation are the same as the x -intercepts for the graph, or the point(s) at which the graph intersects the x -axis. This should make sense as quadratic standard form is in terms of x and is equal to zero. In essence, when we solve a quadratic equation in quadratic standard form, we are solving for x -values that make the relation — the y -coordinate — equal to zero. This is the definition of the x -intercept. Begin to introduce students to this concept in preparation for Part C simply by associating the terms "root," "zero," "solution," and " x -intercept" together as synonyms when dealing with quadratics.

Lesson 12: One Coefficient Is Zero

OBJECTIVE: Solve an equation in quadratic standard form in which one of the coefficients is zero.

IMPORTANT TERMS

1. **Quadratic Relation:** Another name for a second-degree relation. "Quadratic" derives from the Latin *quadratus*, meaning "made square."
2. **Quadratic Standard Form:** A second-degree polynomial in the form $ax^2 + bx + c = 0$ where a , b , and c are real numbers and $a \neq 0$.
3. **Roots of a Quadratic Relation:** The solutions, or x -intercepts, of the relation when the relation equals zero, which is why roots are also called zeros.

EXAMPLE 1

Convert each of the following relations to quadratic standard form, identify a , b , and c , and solve.

- a. $7x + 3 = 15$ b. $3x^2 = 6$ c. $2x^2 + 5x - 3 = -3$

Solution

Begin by converting each relation to quadratic standard form. If the relation is second degree, you will need to solve by the general factoring strategy and the zero-product property.

- a. $7x + 3 = 15$
 $7x + 3 + (-15) = 15 + (-15)$
 $7x - 12 = 0$

In quadratic standard form, $a = 0$, $b = 7$, and $c = -12$. Since $a = 0$, this relation is not quadratic; rather, it is first degree. Solve it by the addition and multiplication properties of equality.

$$\begin{aligned}7x - 12 &= 0 \\7x - 12 + 12 &= 0 + 12 \\7x &= 12 \\\frac{1}{7}(7x) &= \frac{1}{7}(12) \\x &= \frac{12}{7} \\S &= \left\{ \frac{12}{7} \right\}\end{aligned}$$

Remember, a cannot equal zero in quadratic standard form. This is a first degree, or linear, relation.

- b. $3x^2 = 6$
 $3x^2 + (-6) = 6 + (-6)$
 $3x^2 - 6 = 0$

In this case, $a = 3$, $b = 0$, and $c = -6$. Factor this quadratic relation into linear factors by the general factoring strategy.

EXAMPLE 1 (continued)

First, factor out the common factor of 3.

$$3x^2 - 6 = 0$$

$$\textbf{Factor} \quad (3)(x^2 - 2) = 0$$

Next, factor the quadratic binomial as a difference of two squares. Use radicals because 2 is not a perfect square.

$$(3)(x^2 - 2) = 0$$

$$\textbf{Factor} \quad (3)(x + \sqrt{2})(x - \sqrt{2}) = 0$$

Set each linear factor equal to zero and solve for x per the zero-product property.

$$\begin{array}{ccc} (3) & (x + \sqrt{2}) & (x - \sqrt{2}) \\ 3 \neq 0 & x + \sqrt{2} = 0 & x - \sqrt{2} = 0 \\ x + \sqrt{2} + (-\sqrt{2}) = 0 + (-\sqrt{2}) & x - \sqrt{2} + \sqrt{2} = 0 + \sqrt{2} & \\ x = -\sqrt{2} & x = \sqrt{2} & \end{array}$$

The solution set for this relation is $S = \{\pm\sqrt{2}\}$.

We can generalize based on this result. When $b = 0$, rather than converting to quadratic standard form, isolate x and use the principle of powers with a rational exponent that is the reciprocal of the degree of x . Remember the $\pm$ when taking the square root of both sides.

$$\begin{array}{l} 3x^2 = 6 \\ \frac{1}{3}(3x^2) = \frac{1}{3}(6) \\ x^2 = 2 \\ (x^2)^{\frac{1}{2}} = 2^{\frac{1}{2}} \quad \text{or} \quad \sqrt{x^2} = \pm\sqrt{2} \\ x = \pm\sqrt{2} \end{array}$$

The solution set for this relation is $S = \{\pm\sqrt{2}\}$

c. $2x^2 + 5x - 3 = -3$

$$2x^2 + 5x - 3 + 3 = -3 + 3$$

$$2x^2 + 5x = 0$$

In this case $a = 2$, $b = 5$, and $c = 0$. Since this relation is second degree, factor by the general factoring strategy. Begin by factoring out the common factor x . Then use the zero-product property to solve.

$$2x^2 + 5x = 0$$

$$\textbf{Factor} \quad (x)(2x + 5) = 0$$

$$\begin{array}{l} x = 0 \\ 2x + 5 = 0 \\ 2x + 5 + (-5) = 0 + (-5) \\ 2x = -5 \\ \frac{1}{2}(2x) = \frac{1}{2}(-5) \\ x = -\frac{5}{2} \end{array}$$

EXERCISE KEY

1. $S = \{-9, 9\}$
2. $S = \{18\}$
3. $S = \{0, -3\}$
4. $S = \{-5, 5\}$
5. $S = \{-4, 4\}$
6. $S = \{6\}$
7. $S = \{-\sqrt{5}, \sqrt{5}\}$
8. $S = \left\{\frac{12}{11}\right\}$
9. $S = \{0, 1\}$
10. $S = \{0, -5\}$
11. $S = \{6, -6\}$
12. $S = \{24\}$
13. $S = \{0, 2\}$
14. $S = \{6, -6\}$
15. $S = \left\{\frac{2}{5}\right\}$
16. $S = \{\sqrt{3}, -\sqrt{3}\}$

EXAMPLE 1 (continued)

The solution set is $S = \left\{0, \frac{-5}{2}\right\}$.

We can generalize based on this result as well. When $c = 0$, one of the solutions will always be zero.

EXERCISES

Convert each of the following relations to quadratic standard form and identify a , b , and c . Then solve for x and state the solution set in set notation.

- | | | |
|---------------------------|-------------------------|-----------------------|
| 1. $p^2 - 81 = 0$ | 7. $x^2 + 1 = 6$ | 13. $9z^2 - 3z = 15z$ |
| 2. $\frac{2}{3}y - 5 = 7$ | 8. $7x - 9 = 3 - 4x$ | 14. $x^2 + 6 = 42$ |
| 3. $x^2 + 3x = 0$ | 9. $m^2 = m$ | 15. $2y + 3y = 2$ |
| 4. $t^2 = 25$ | 10. $4y^2 + 20y = 0$ | 16. $5r^2 - 15 = 0$ |
| 5. $-9z^2 + 144 = 0$ | 11. $3x^2 = 108$ | |
| 6. $9x - 12 = 7x$ | 12. $\frac{5}{8}b = 15$ | |

LESSON 24

The purpose of this lesson is twofold: to learn how to convert quadratic functions into vertex form so they can be graphed quickly, and to continue to practice graphing quadratic functions quickly and stating their defining characteristics.

In order to convert quadratic functions into vertex form, you will teach students a variation on completing the square.

Example 1 demonstrates this process in detail. The important caveat here is that rather than adding a constant term to both sides of the equation, you will add two constant terms to the same side of the equation. Therefore, the two terms must be additive inverses of each other so that their sum is zero. Furthermore, pay special attention to the a value (in Example 1, it is 2). If you add a constant term inside the group to complete the trinomial square, you must add the opposite of that number multiplied by the a value to the same side of the equation. This is why, in Example 1, 4 is added to complete the square but -8 is added outside the group to balance the equation: $2(-4) = -8$.

Students should be growing in their proficiency regarding graphing quadratic functions quickly from vertex form and stating their defining characteristics.

Lesson 24: Graphing Quadratic Functions — Converting to Vertex Form

OBJECTIVE: Convert quadratic functions into vertex form to graph them quickly.

IMPORTANT TERMS

- Vertex Form:** A standard form for quadratic functions that communicates the location of the vertex and the orientation of the parabola. General vertex form is $f(x) = a(x - h)^2 + k$.
- Completing the Square:** A solution methodology in which an unfactorable equation in quadratic standard form is transformed into a trinomial square so that it can be solved.

EXAMPLE 1

Graph the quadratic function $y = 2x^2 + 8x + 1$ and state its defining characteristics.

Solution

The best way to quickly graph quadratic functions that are not in vertex form is to utilize an application of completing the square to convert the function into vertex form. Once in vertex form, you can identify the characteristics and graph it quickly.

Begin by factoring the coefficient of the second-degree term (the a value) out of the first two terms as follows:

$$y = 2(x^2 + 4x) + 1$$

Then, complete the square inside the group. Since you are adding a term to the right side of the equation, you must also add its opposite to the right side so that the change is zero.

$$y = 2(x^2 + 4x + \underline{\quad}) + 1 + \underline{\quad}$$

$$y = 2(x^2 + 4x + 4) + 1 + \underline{2(-4)}$$

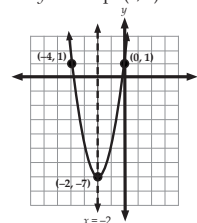
$$y = 2(x^2 + 4x + 4) - 7$$

Factor the resulting trinomial square in the group into a square binomial. The result is vertex form.

$$y = 2(x + 2)^2 - 7$$

Identify the characteristics of the function. The vertex is the point $(-2, -7)$, the axis of symmetry is $x = -2$, it opens upward with a vertical stretch, and the minimum value is -7 .

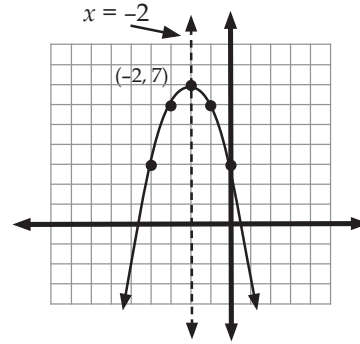
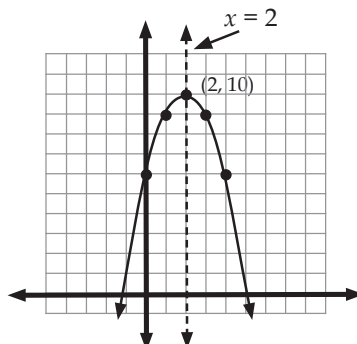
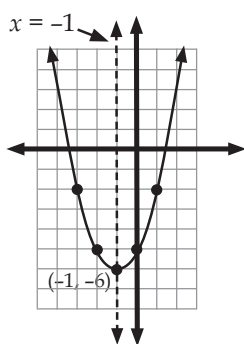
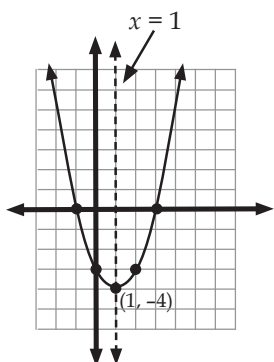
Now, graph the function quickly. Use the y -intercept $(0, 1)$ and its symmetrical point $(-4, 1)$ as shown.



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EXERCISE KEY

- $f(x) = (x - 1)^2 - 4$
- $f(x) = (x + 1)^2 - 6$
- $f(x) = -1(x - 2)^2 + 10$
- $f(x) = -(x + 2)^2 + 7$

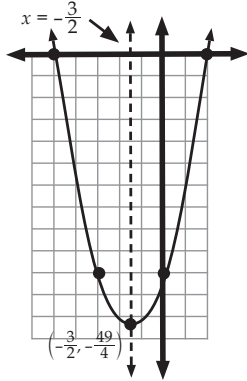


EXERCISES

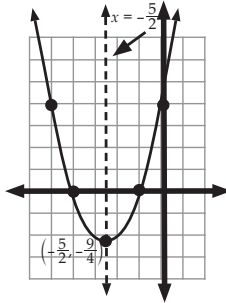
Sketch the graph of each of the following functions and state its defining characteristics.

1. $f(x) = x^2 - 2x - 3$
2. $f(x) = x^2 + 2x - 5$
3. $f(x) = -x^2 + 4x + 6$
4. $f(x) = -x^2 - 4x + 3$
5. $f(x) = x^2 + 3x - 10$
6. $f(x) = x^2 + 5x + 4$
7. $f(x) = x^2 - 9x$
8. $f(x) = x^2 + x$
9. $f(x) = 3x^2 - 24x + 50$
10. $f(x) = 4x^2 + 8x - 3$
11. $f(x) = \frac{3}{4}x^2 + 9x$
12. $f(x) = x^2 - 6x + 4$
13. $f(x) = x^2 - 4x + 7$
14. $f(x) = -2x^2 + 10x - 7$
15. $f(x) = -4x^2 + 12x - 5$

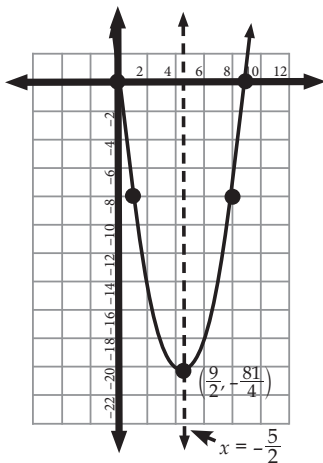
5. $f(x) = \left(x + \frac{3}{2}\right)^2 - \frac{49}{4}$



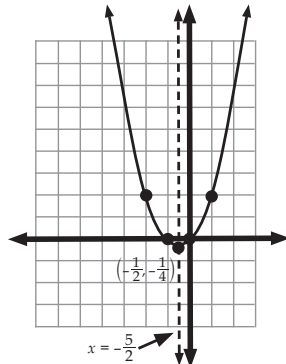
6. $f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{9}{4}$



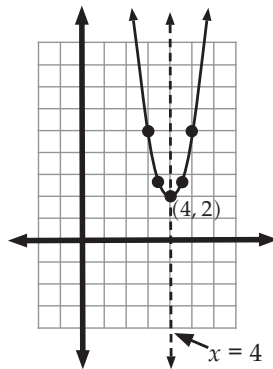
7. $f(x) = \left(x - \frac{9}{2}\right)^2 - \frac{81}{4}$



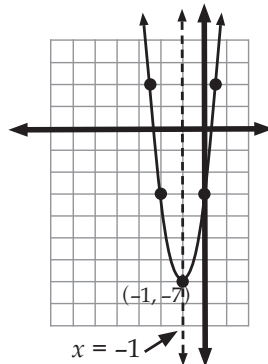
8. $f(x) = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$



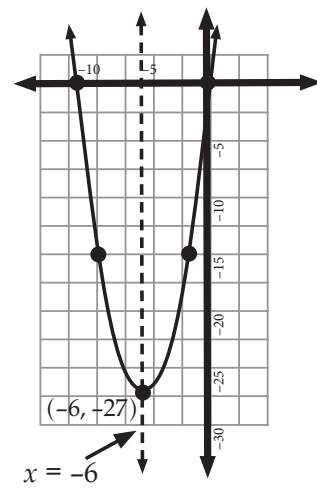
9. $f(x) = 3(x - 4)^2 + 2$



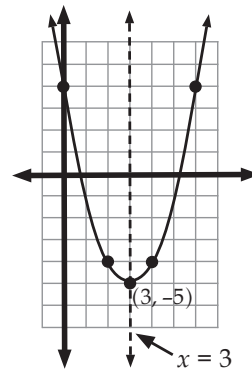
10. $f(x) = 4(x + 1)^2 - 7$



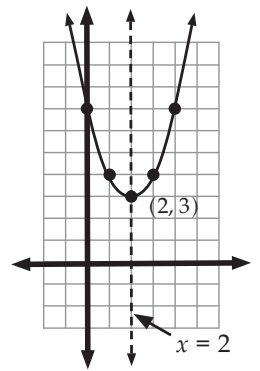
11. $f(x) = \frac{3}{4}(x + 6)^2 - 27$



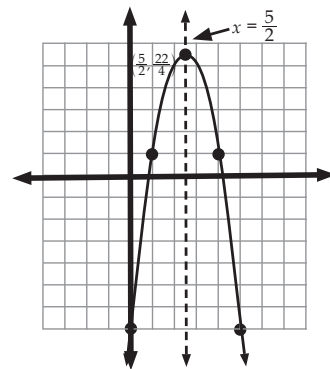
12. $f(x) = (x - 3)^2 - 5$



13. $f(x) = (x - 2)^2 + 3$



14. $f(x) = -2\left(x - \frac{5}{2}\right)^2 + \frac{22}{4}$



15. $f(x) = -4\left(x - \frac{3}{2}\right)^2 + 4$

